SPIN-2002/40

A discrete history of the Lorentzian path integral

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January 3, 2003

Abstract

In these lecture notes, I describe the motivation behind a recent formulation of a non-perturbative gravitational path integral for Lorentzian (instead of the usual Euclidean) space-times, and give a pedagogical introduction to its main features. At the regularized, discrete level this approach solves the problems of (i) having a well-defined Wick rotation, (ii) possessing a coordinate-invariant cutoff, and (iii) leading to *convergent* sums over geometries. Although little is known as yet about the existence and nature of an underlying continuum theory of quantum gravity in four dimensions, there are already a number of beautiful results in $d = 2$ and $d = 3$ where continuum limits have been found. They include an explicit example of the inequivalence of the Euclidean and Lorentzian path integrals, a non-perturbative mechanism for the cancellation of the conformal factor, and the discovery that causality can act as an effective regulator of quantum geometry.

1 Introduction

The desire to understand the quantum physics of the gravitational interactions lies at the root of many recent developments in theoretical high-energy physics. By *quantum gravity* I will mean a consistent fundamental quantum description of space-time geometry (with or without matter) whose classical limit is general relativity. Among the possible ramifications of such a theory are a model for the structure of space-time near the Planck scale, a consistent calculational scheme to compute gravitational effects at all energies, a description of (quantum) geometry near space-time singularities and a non-perturbative quantum description of four-dimensional black holes. It might also help us in understanding cosmological issues about the beginning (and end?) of our universe, although it should be said that some questions (for example, that of the "initial conditions") are likely to remain outside the scope of any physical theory.

From what we know about the quantum dynamics of the other fundamental interactions it seems eminently plausible that also the gravitational excitations should at very short scales be governed by quantum laws, so why have we so far not been able to determine what they are? – One obvious obstacle is the difficulty in finding any direct or indirect evidence for quantum gravitational effects, be they experimental or observational, which could provide a feedback for model-building. A theoretical complication is that the outstanding problems mentioned above require a non-perturbative treatment; it is not sufficient to know the first few terms of a perturbation series. This is true for both conventional perturbative path integral expansions of gravity or supergravity[1] and a perturbative expansion in the string coupling in the case of unified approaches. One avenue to take is to search for a *non-perturbative* definition of such a theory, where the initial input of any fixed "background metric" is inessential (or even undesirable), and where "space-time" is determined *dynamically*. Whether or not such an approach necessarily requires the inclusion of higher dimensions and fundamental supersymmetry is currently unknown. As we will see in the course of these lecture notes, it is perfectly conceivable that one can do without.

Such a non-perturbative viewpoint is very much in line with how one proceeds in classical general relativity, where a metric space-time $(M, g_{\mu\nu})$ (+matter) emerges only as a *solution* to the Einstein equations

$$R_{\mu\nu}[g] - \frac{1}{2} g_{\mu\nu} R[g] + \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}[\Phi], \tag{1}$$

which define the classical dynamics on the space $\mathcal{M}(M)$, the space of all metrics on a given differentiable manifold M. The analogous question I want to address in the quantum theory is

> Can we obtain "quantum space-time" as a solution to a set of non-perturbative quantum equations of motion on a suitable quantum analogue of $\mathcal{M}(M)$ or rather, of the space of geometries, $\mathrm{Geom}(M) := \mathcal{M}(M)/\mathrm{Diff}(M)$?

[1]Of course, we already know that in these cases a quantization based on a decomposition $g_{\mu\nu}(x) = \eta^{\rm Mink}_{\mu\nu} + \sqrt{G_N}\, h_{\mu\nu}(x)$, for a linear spin-2 perturbation around Minkowski space leads to a non-renormalizable theory.

This is not a completely straightforward task. Whichever way we want to proceed non-perturbatively, if we give up the privileged role of a flat, Minkowskian background space-time on which the quantization is to take place, we also have to abandon the central role usually played by the Poincaré group, and with it most standard quantum field-theoretic tools for regularization and renormalization. If one works in a continuum metric formulation of gravity, the symmetry group of the Einstein action is instead the group Diff(M) of diffeomorphisms on M, which in terms of local charts are simply the smooth coordinate transformations $x^\mu \mapsto y^\mu(x^\mu)$.[2]

I will in the following describe a particular path integral approach to quantum gravity, which is non-perturbative from the outset in the sense of being defined on the "space of all geometries" (to be defined later), without distinguishing any background metric structure (see also [1, 2] for related reviews). This is closely related in spirit with the canonical approach of loop quantum gravity [3] and its more recent incarnations using so-called spin networks [4, 5], although there are significant differences in methodology and attitude. "Non-perturbative" means in a covariant context that the path sum or integral will have to be performed explicitly, and not just evaluated around its stationary points, which can only be achieved in an appropriate regularization. The method I will employ uses a discrete lattice regularization as an intermediate step in the construction of the quantum theory. However, unlike in lattice QCD, the lattice and its geometric properties will not be part of a static background structure, but dynamical quantities, as befits a theory of *quantum geometry*.

2 Quantum gravity from dynamical triangulations

In this section I will explain how one may construct a theory of quantum gravity from a non-perturbative path integral, and what logic has led my collaborators and me to consider the method of Lorentzian dynamical triangulations to achieve this. The method is minimal in the sense of employing standard tools from quantum field theory and the theory of critical phenomena and adapting them to the case of *generally covariant systems*, without invoking any symmetries beyond those of the classical theory. At an intermediate stage of the construction, we use a regularization in terms of simplicial "Regge geometries", that is, piecewise linear manifolds. In this approach, "computing the path integral" amounts to a conceptually simple and geometrically transparent

[2]One should not get confused here by the fact that in gauge formulations of gravity which work with vierbeins e^a_μ instead of the metric tensor $g_{\mu\nu}$, one has an additional local invariance under SO(3,1)-frame rotations, ie. elements of the Lorentz group, in addition to diffeomorphism invariance. Nevertheless, this formulation is still not invariant under *global* Lorentz- or Poincaré transformations.

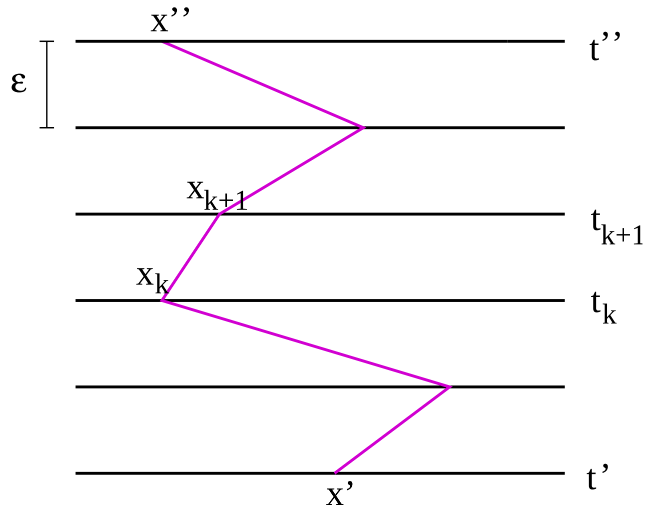

Figure 1: A piecewise linear particle path contributing to the discrete Feynman propagator.

"counting of geometries", with additional weight factors which are determined by the Einstein action. This is done first of all at a regularized level. Subsequently, one searches for interesting continuum limits of these discrete models which are possible candidates for theories of quantum gravity, a step that will always involve a renormalization. From the point of view of statistical mechanics, one may think of Lorentzian dynamical triangulations as a new class of statistical models of Lorentzian random surfaces in various dimensions, whose building blocks are flat simplices which carry a "time arrow", and whose dynamics is entirely governed by their intrinsic geometric properties.

Before describing the details of the construction, it may be helpful to recall the path integral representation for a (one-dimensional) non-relativistic particle [6]. The time evolution of the particle's wave function ψ may be described by the integral equation

$$\psi(x'',t'') = \int_{\mathbf{R}} G(x'',x';t'',t')\psi(x',t'), \tag{2}$$

where the propagator or Feynman kernel G is defined through a limiting procedure,

$$G(x'',x';t'',t') = \lim_{\epsilon\longrightarrow 0} A^{-N} \prod_{k=1}^{N-1} \int dx_k \ \mathrm{e}^{i \sum_{j=0}^{N-1} \epsilon L(x_{j+1},(x_{j+1}-x_j)/\epsilon)}. \tag{3}$$

The time interval $t''-t'$ has been discretized into N steps of length $\epsilon = (t''-t')/N$, and the right-hand side of (3) represents an integral over all piecewise linear paths $x(t)$ of a "virtual" particle propagating from x' to x'', illustrated in Fig.1.

The prefactor A^{-N} is a normalization and L denotes the Lagrange function of the particle. Knowing the propagator G is tantamount to having solved the quantum dynamics. This is the simplest instance of a *path integral*, and is often written schematically as

$$G(x', t'; x'', t'') = \int \mathcal{D}x(t)\ \mathrm{e}^{iS[x(t)]}, \tag{4}$$

where $\mathcal{D}x(t)$ is a functional measure on the "space of all paths", and the exponential weight depends on the classical action $S[x(t)]$ of a path. Recall also that this procedure can be defined in a mathematically clean way if we wick-rotate the time variable t to imaginary values $t \mapsto \tau = it$, thereby making all integrals real [7].

Can a similar strategy work for the case of Einstein gravity? As an analogue of the particle's position we can take the geometry $[g_{ij}(x)]$ (ie. an equivalence class of spatial metrics) of a constant-time slice. Can one then define a gravitational propagator

$$G([g'_{ij}], [g''_{ij}]) = \int_{\mathrm{Geom(M)}} \mathcal{D}[g_{\mu\nu}]\ \mathrm{e}^{iS^{\mathrm{Einstein}}[g_{\mu\nu}]} \tag{5}$$

from an initial geometry $[g']$ to a final geometry $[g'']$ (Fig.2 as a limit of some discrete construction analogous to that of the non-relativistic particle (3)? And crucially, what would be a suitable class of "paths", that is, space-times $[g_{\mu\nu}]$ to sum over?

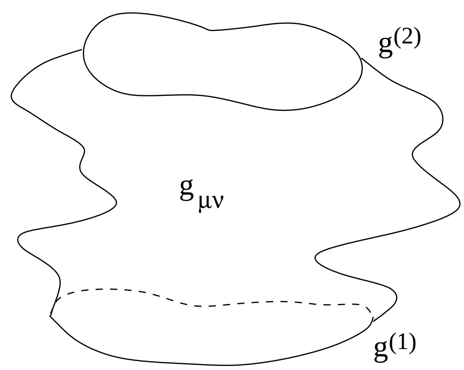

Figure 2: The time-honoured way [8] of illustrating the gravitational path integral from an initial to a final spatial boundary geometry.

Setting aside the question of the physical meaning of an expression like (5), gravitational path integrals in the continuum are extremely ill-defined. Clearly, *defining* a fundamental theory of quantum gravity via a perturbation series in the gravitational coupling does not work because of its perturbative non-renormalizability. So, is there a chance we might simply be able to *do* the integration $\int \mathcal{D}[g_{\mu\nu}]$ in a meaningful way? Firstly, there is no obvious way to parametrize "geometries", which means that in practice one always has to start with gauge-*co*variant fields, and gauge-fix. Unfortunately, this gives rise to Faddeev-Popov determinants whose non-perturbative evaluation is exceedingly difficult. A similar problem already applies to the action itself, which is by no means quadratic, no matter what we choose as our basic fields. How then can the integration over $\exp(iS)$ possibly be performed? Part of the problem is clearly also the complex nature of this integrand, with no obvious choice of a Wick rotation in the context of a theory with fluctuating geometric degrees of freedom. Secondly, since we are dealing with a field theory, some kind of regularization will be necessary, and the challenge here is to find a procedure that does not violate diffeomorphism-invariance.

In brief, the strategy I will be following starts from a regularized version of the space $\mathrm{Geom}(M)$ of all geometries. A regularized path integral $G(a)$ can be defined which depends on an ultraviolet cutoff a and is *convergent* in a non-trivial region of the space of coupling constants. Taking the continuum limit corresponds to letting $a \to 0$. The resulting continuum theory – if it can be shown to exist – is then investigated with regard to its geometric properties and in particular its semiclassical limit.

3 Brief summary of discrete gravitational path integrals

Trying to construct non-perturbative path integrals for gravity from sums over discretized geometries is not a new idea. The approach of *Lorentzian dynamical triangulations* draws from older work in this area, but differs from it in several significant aspects as we shall see in due course.

Inspired by the successes of lattice gauge theory, attempts to describe quantum gravity by similar methods have been popular on and off since the late 70's. Initially the emphasis was on gauge-theoretic, first-order formulations of gravity, usually based on (compactified versions of) the Lorentz group, followed in the 80's by "quantum Regge calculus", an attempt to represent the gravitational path integral as an integral over certain piecewise linear geometries (see [9] and references therein), which had first made an appearance in approximate descriptions of *classical* solutions of the Einstein equations. A variant of this approach by the name of "dynamical triangulation(s)" attracted a lot of interest during the 90's, partly because it had proved a powerful tool in describing two-dimensional quantum gravity (see the textbook [10] and lecture notes [11] for more details).

The problem is that none of these attempts have so far come up with convincing evidence for the existence of an underlying continuum theory of four-dimensional quantum gravity. This conclusion is drawn largely on the basis of numerical simulations, so it is by no means water-tight, although one can make an argument that the "symptoms" of failure are related in the various approaches [12]. What goes wrong generically seems to be a dominance in the continuum limit of highly degenerate geometries, whose precise form depends on the approach chosen. One would of course expect that non-smooth geometries play a decisive role, in the same way as it can be shown in the particle case that the support of the measure in the continuum limit is on a set of nowhere differentiable paths. However, what seems to happen in the case of the path integral for four-geometries is that the structures obtained are *too* wild, in the sense of not generating, even at coarse-grained scales, an effective geometry whose dimension is anywhere near four.

The schematic phase diagram of Euclidean dynamical triangulations shown in Fig.3 gives an example of what can happen. The picture turns out to be essentially the same in both three and four dimensions: the model possesses infinite-volume limits everywhere along the critical line $k_3^{\rm crit}(k_0)$, which fixes the bare cosmological constant as a function of the inverse Newton constant $k_0 \sim G_N^{-1}$. Along this line, there is a critical point $k_0^{\rm crit}$ (which we now know to be of first order in $d = 3, 4$) below which geometries generically have a very large effective or Hausdorff dimension (there are a few vertices at which the entire space-time "condenses" in the sense that almost every other vertex in the simplicial space-time is about one link-distance away from them). Above $k_0^{\rm crit}$ we find the opposite phenomenon of "polymerization": a typical element contributing to the state sum is a thin branched polymer, with one or more dimensions "curled up" (an image familiar to string theorists!) such that its effective dimension is around two.

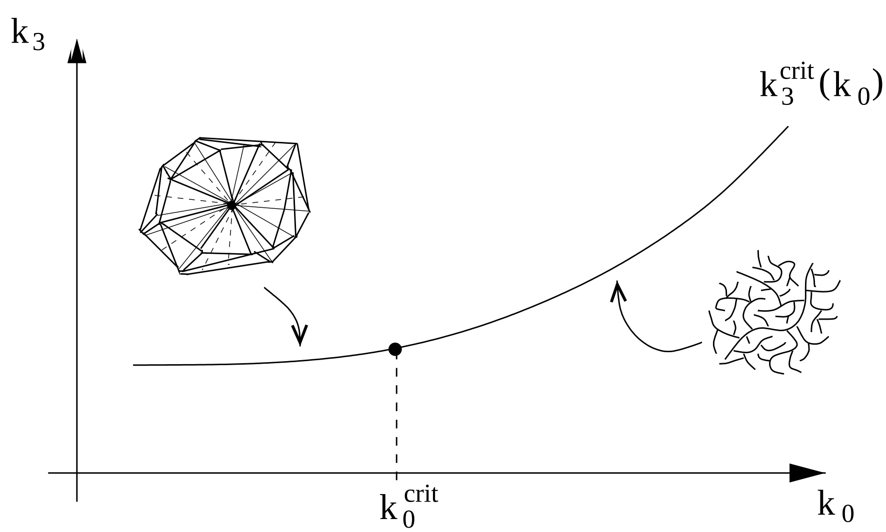

Figure 3: The phase diagram of three- and four-dimensional Euclidean dynamical triangulations.

Why this happens was, at least until recently, less clear, although it has sometimes be related to the so-called conformal-factor problem. This problem has to do with the fact that the gravitational action is unbounded below, causing potential havoc in Euclidean versions of the path integral. This will be discussed in more detail below in Sec. 5.2, but it does lead directly to the next point. Namely, what all the above-mentioned approaches have in common is that they work from the outset with *Euclidean* geometries, and associated Boltzmann-type weights $\exp(-S^{\mathrm{eu}})$ in the path integral. In other words, they integrate over "space-times" which know nothing about time, light cones and causality. This is done mainly for technical reasons, since it is difficult to set up simulations with complex weights and since until recently a suitable Wick rotation was not known.

"Lorentzian dynamical triangulations", first proposed in [13] and further elaborated in [14, 15] tries to establish a logical connection between the fact that all non-perturbative path integrals were constructed for Euclidean instead of Lorentzian geometries and their apparent failure to lead to an interesting continuum theory. Is it conceivable that we can kill two birds with one stone, ie. cure the problem of degenerate quantum geometry by taking a path integral over geometries with a physical, Lorentzian signature? Remarkably, this is indeed what happens in the quantum gravity theories in $d < 4$ which have already been studied extensively. The way in which Lorentzian dynamical triangulations overcome the problems mentioned above is the subject of the Sec. 5.

4 Geometry from simplices

The use of simplicial methods in general relativity goes back to the pioneering work of Regge [16]. In classical applications one tries to approximate a classical space-time geometry by a triangulation, that is, a piecewise linear space obtained by gluing together flat simplicial building blocks, which in dimension d are d-dimensional generalizations of triangles. By "flat" I mean that they are isometric to a subspace of d-dimensional Euclidean or Minkowski space. We will only be interested in gluings leading to genuine manifolds, which locally look like an R^d. A nice feature of such simplicial manifolds is that their geometric properties are completely described by the discrete set $\{l_i^2\}$ of the squared lengths of their edges. Note that this amounts to a description of geometry *without the use of coordinates.* There is nothing to prevent us from re-introducing coordinate patches covering the piecewise linear manifold, for example, on each individual simplex, with suitable transition functions between patches. In such a coordinate system the metric tensor will take on a definite form. However, for the purposes of formulating the path integral we will not be interested in doing this, but rather work with the edge lengths, which constitute a direct, regularized parametrization of the

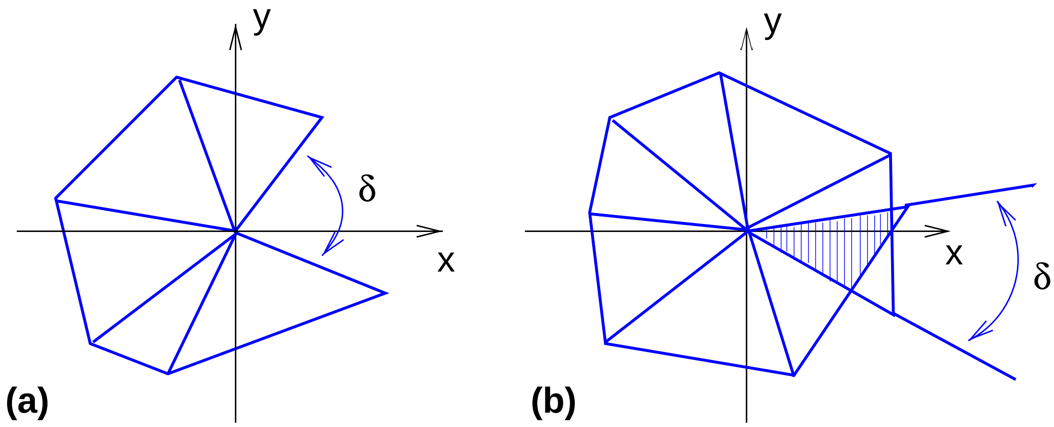

Figure 4: Positive (a) and negative (b) space-like deficit angles δ.

space Geom(M) of geometries.

How precisely is the intrinsic geometry of a simplicial space, most importantly, its curvature, encoded in its edge lengths? A useful example to keep in mind is the case of dimension two, which can easily be visualized. A 2d piecewise linear space is a triangulation, and its scalar curvature $R(x)$ coincides with the so-called Gaussian curvature. One way of measuring this curvature is by parallel-transporting a vector around closed curves in the manifold. In our piecewise-flat manifold such a vector will always return to its original orientation *unless* it has surrounded lattice vertices v at which the surrounding angles did not add up to 2π, but $\sum_{i\supset v}\alpha_i = 2\pi-\delta$, for $\delta\neq 0$, see Fig.4. The so-called deficit angle δ is precisely the rotation angle picked up by the vector and is a direct measure for the scalar curvature at the vertex. The operational description to obtain the scalar curvature in higher dimensions is very similar, one basically has to sum in each point over the Gaussian curvatures of all two-dimensional submanifolds. This explains why in Regge calculus the curvature part of the Einstein action is given by a sum over building blocks of dimension $(d-2)$ which are simply the objects dual to those local 2d submanifolds. More precisely, the continuum curvature and volume terms of the action become

$$\frac{1}{2}\int_{\mathcal{R}} d^dx\,\sqrt{|\det g|}^{(d)}R \quad\longrightarrow\quad \sum_{i\in\mathcal{R}} Vol(i^{th}\ (d-2)\mathrm{-simplex})\ \delta_i \tag{6}$$

$$\int_{\mathcal{R}} d^dx\,\sqrt{|\det g|} \quad\longrightarrow\quad \sum_{i\in\mathcal{R}} Vol(i^{th}\ d\mathrm{-simplex}) \tag{7}$$

in the simplicial discretization. It is then a simple exercise in trigonometry to express the volumes and angles appearing in these formulas as functions of the edge lengths l_i, both in the Euclidean and Minkowskian case.

The approach of dynamical triangulations uses a certain class of such simplicial space-times as an explicit, regularized realization of the space Geom(M). For a given volume N_d, this class consists of all gluings of manifold-type of a set of N_d simplicial building blocks of top-dimension d whose edge lengths are restricted to take either one or one out of two values. In the Euclidean case we set $l_i^2 = a^2$ for all i, and in the Lorentzian case we allow for both space- and time-like links with $l_i^2 \in \{-a^2, a^2\}$, where the geodesic distance a serves as a short-distance cutoff, which will be taken to zero later. Coming from the classical theory this may seem a grave restriction at first, but this is indeed not the case. Firstly, keep in mind that for the purposes of the quantum theory we want to sample the space of geometries "ergodically" at a coarse-grained scale of order a. This should be contrasted with the classical theory where the objective is usually to approximate a given, *fixed* space-time to within a length scale a. In the latter case one typically requires a much finer topology on the space of metrics or geometries. It is also straightforward to see that no local curvature degrees of freedom are suppressed by fixing the edge lengths; deficit angles in all directions are still present, although they take on only a discretized set of values. In this sense, in dynamical triangulations all geometry is in the gluing of the fundamental building blocks. This is dual to what how quantum Regge calculus is set up, where one usually fixes a triangulation T and then "scans" the space of geometries by letting the l_i's run continuously over all values compatible with the triangular inequalities.

In a nutshell, Lorentzian dynamical triangulations give a definite meaning to the "integral over geometries", namely, as a sum over inequivalent Lorentzian gluings T over any number N_d of d-simplices,

$$\int_{\mathrm{Geom(M)}} \mathcal{D}[g_{\mu\nu}]\; \mathrm{e}^{iS[g_{\mu\nu}]} \quad \stackrel{\mathrm{LDT}}{\longrightarrow} \quad \sum_{T\in\mathcal{T}} \frac{1}{C_T}\; \mathrm{e}^{iS^{\mathrm{Regge}}(T)}, \tag{8}$$

where the symmetry factor $C_T = |Aut(T)|$ on the right-hand side is the order of the automorphism group of the triangulation, consisting of all maps of T onto itself which preserve the connectivity of the simplicial lattice. I will specify below what precise class $\mathcal{T}$ of triangulations should appear in the summation.

It follows from the above that in this formulation all curvatures and volumes contributing to the simplicial Regge action come in discrete units. This is again easily illustrated by the case of a two-dimensional triangulation with Euclidean signature, which according to the prescription of dynamical triangulations consists of *equilateral* triangles with squared edge lengths $+a^2$. All interior angles of such a triangle are equal to $\pi/3$, which implies that the deficit angle at any vertex v can take the values $2\pi - k_v\pi/3$, where k_v is the number of triangles meeting at v. As a consequence, the

Einstein-Regge action assumes the simple form[3]

$$S^{\rm Regge}(T) = \kappa_{d-2} N_{d-2} - \kappa_d N_d, \tag{9}$$

where the coupling constants $\kappa_i = \kappa_i(\lambda, G_N)$ are simple functions of the bare cosmological and Newton constants in d dimensions. Substituting this into the path sum in (8) leads to

$$Z(\kappa_{d-2}, \kappa_d) = \sum_{N_d} {\rm e}^{-i\kappa_d N_d} \sum_{N_{d-2}} {\rm e}^{i\kappa_{d-2} N_{d-2}} \sum_{T|_{N_d, N_{d-2}}} \frac{1}{C_T}, \tag{10}$$

The point of taking separate sums over the numbers of d- and $(d-2)$-simplices in (10) is to make explicit that "doing the sum" is tantamount to the combinatorial problem of *counting* triangulations of a given volume and number of simplices of co-dimension two (corresponding to the last summation in (10)). It turns out that at least in two space-time dimensions the counting of geometries can be done completely explicitly, turning both Lorentzian and Euclidean quantum gravity into exactly soluble statistical models.

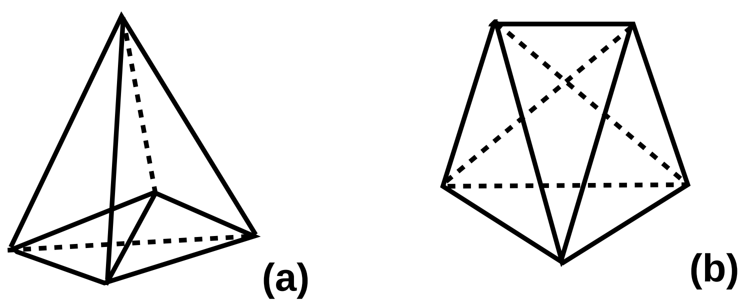

Figure 5: The two types of Minkowskian four-simplices in four dimensions.

5 Lorentzian nature of the path integral

It is now time to explain what makes our approach *Lorentzian* and why it therefore differs from previous attempts at constructing non-perturbative gravitational path integrals. The simplicial building blocks of the models are taken to be pieces of Minkowski space, and their edges have squared lengths $+a^2$ or $-a^2$. For example, the two types of

[3]Strictly speaking, the expression (9) in $d \geq 3$ is only correct for the Euclidean or the wick-rotated Lorentzian action. In the Lorentzian case one has several types of simplices of a given dimension d, depending on how many of its links are time-like. Only after the Wick rotation will all links be space-like and of equal length (see later). Nevertheless, I will use this more compact form for ease of notation.

four-simplices that are used in Lorentzian dynamical triangulations in dimension four are shown in Fig.5. The first of them has four time-like and six space-like links (and therefore contains 4 time-like and 1 space-like tetrahedron), whereas the second one has six time-like and four space-like links (and contains 5 time-like tetrahedra). Since both are subspaces of flat space with signature $(-+++)$, they possess well defined light-cone structures everywhere.

In general, gluings between pairs of d-simplices are only possible when the metric properties of their (d-1)-faces match. Having local light cones implies causal relations between pairs of points in local neighbourhoods.Creating closed time-like curves will be avoided by requiring that all space-times contributing to the path sum possess a global "time" function t. In terms of the triangulation this means that the d-simplices are arranged such that their space-like links all lie in slices of constant integer t, and their time-like links interpolate between adjacent spatial slices t and $t+1$. Moreover, with respect to this time, we will not allow for any *spatial* topology changes[4].

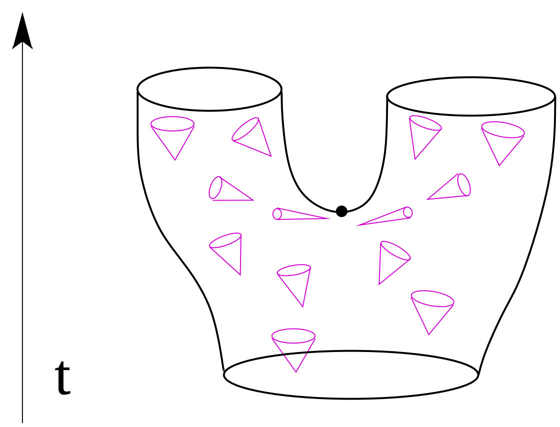

Figure 6: At a branching point associated with a spatial topology change, light-cones get "squeezed".

This latter condition is always satisfied in classical applications, where "trouser points" like the one depicted in Fig.6 are ruled out by the requirement of having a non-degenerate Lorentzian metric defined everywhere on M (it is geometrically obvious that the light cone and hence $g_{\mu\nu}$ must degenerate in at least one point along the "crotch"). Another way of thinking about such configurations (and their time-reversed counterparts) is that the causal past (future) of an observer changes discontinuously as she passes near the singular point.

[4]Note that if we were in the continuum and had introduced coordinates on space-time, such a statement would actually be diffeomorphism-invariant.

Of course, there is no *a priori* reason in the quantum theory to not relax some of these classical causality constraints. After all, as I stressed right at the outset, path integral histories are not in general classical solutions, nor can we attribute any other direct physical meaning to them individually. It might well be that one can construct models whose path integral configurations violate causality in this strict sense, but where it is somehow recovered in the resulting continuum theory. What the approach of Lorentzian dynamical triangulations has demonstrated is that *imposing causality constraints will in general lead to a different continuum theory.* This is in contrast with the intuition one may have that "including a few isolated singular points will not make any difference". On the contrary, tampering with causality in this way is not innocent at all, as was already anticipated by Teitelboim many years ago [17].

I want to point out that one cannot conclude from the above that spatial topology changes or even fluctuations in the *space-time* topology cannot be treated in the formulation of dynamical triangulations. However, if one insists on including geometries of variable topology in a Lorentzian discrete context, one has to come up with a prescription of how to weigh these singular points in the path integral, both before and after the Wick rotation. Maybe this can be done along the lines suggested in [18]; this is clearly an interesting issue for further research.

Having said this, we next have to address the question of the Wick rotation, in other words, of how to get rid of the factor of i in the exponent of (10). Without it, this expression is an infinite sum (since the volume can become arbitrarily large) of complex terms whose convergence properties will be very difficult to determine. In this situation, a Wick rotation is simply a technical tool which – in the best of all worlds – enables us to perform the state sum and determine its continuum limit. Of course, the end result will have to be Wick-rotated back to Lorentzian signature.

Fortunately, Lorentzian dynamical triangulations come with a natural notion of Wick rotation, and the strategy I just outlined can be carried out explicitly in two space-time dimensions, leading to a unitary theory (see Sec. 5.1 below). In higher dimensions we do not yet have sufficient analytical control of the continuum theories to make specific statements about the *inverse* Wick rotation. Since we use the Wick rotation at an intermediate step, one can ask whether other Wick rotations would lead to the same result. Currently this is a somewhat academic question, since it is in practice difficult to find such alternatives. In fact, it is quite miraculous we have found a single prescription for Wick rotating, and we are also not aware of any continuum analogues (for more comments on this issue, see [19, 20].

Our Wick rotation W in any dimension is an injective map from Lorentzian- to Euclidean-signature simplicial space-times. Using the notation $\mathcal{T}$ for a simplicial manifold together with length assignments l_s^2 and l_t^2 to its space- and time-like links, it is defined by

$$\mathcal{T}^{\rm lor}=(T,\{l_s^2=a^2,l_t^2=-a^2\})\;\stackrel{W}{\mapsto}\;\mathcal{T}^{\rm eu}=(T,\{l_s^2=a^2,l_t^2=a^2\}).\tag{11}$$

Note that we have not touched the connectivity of the simplicial manifold T, but only its metric properties, by mapping all time-like links of T into space-like ones, resulting in a Euclidean "space-time" of equilateral building blocks. It can be shown [15] that at the level of the corresponding weight factors in the path integral this Wick rotation[5] has precisely the desired effect of rotating to the exponentiated Regge action of the Euclideanized geometry,

$$\mathrm{e}^{iS(\mathcal{T}^{\rm lor})}\;\stackrel{W}{\mapsto}\;\mathrm{e}^{-S(\mathcal{T}^{\rm eu})}.\tag{12}$$

The Euclideanized path sum after the Wick rotation has the form

$$\begin{aligned}Z^{\rm eu}(\kappa_{d-2},\kappa_d) &= \sum_T \frac{1}{C_T}\,\mathrm{e}^{-\kappa_d N_d(T)+\kappa_{d-2}N_{d-2}(T)}\\ &= \sum_{N_d}\mathrm{e}^{-\kappa_d N_d}\sum_{T|_{N_d}}\frac{1}{C_T}\,\mathrm{e}^{\kappa_{d-2}N_{d-2}(T)}\\ &= \sum_{N_d}\mathrm{e}^{-\kappa_d N_d}\,\mathrm{e}^{\kappa_d^{\rm crit}(\kappa_{d-2})N_d}\times \text{subleading}(N_d).\end{aligned}\tag{13}$$

In the last equality I have used that the number of Lorentzian triangulations of discrete volume N_d to leading order scales exponentially with N_d for large volumes. This can be shown explicitly in space-time dimension 2 and 3. For $d=4$, there is strong (numerical) evidence for such an exponential bound for *Euclidean* triangulations, from which the desired result for the Lorentzian case follows (since W maps to a strict subset of all Euclidean simplicial manifolds).

From the functional form of the last line of (13) one can immediately read off some qualitative features of the phase diagram, an example of which appeared already earlier in Fig.3. Namely, the sum over geometries $Z^{\rm eu}$ converges for values $\kappa_d>\kappa_d^{\rm crit}$ of the bare cosmological constant, and diverges (ie. is not defined) below this critical line. Generically, for all models of dynamical triangulations the infinite-volume limit is attained by approaching the critical line $\kappa_d^{\rm crit}(\kappa_{d-2})$ from above, ie. from inside the region of convergence of $Z^{\rm eu}$. In the process of taking $N_d\to\infty$ and the cutoff $a\to 0$, one obtains a renormalized cosmological constant Λ through

$$(\kappa_d-\kappa_d^{\rm crit})=a^\mu\Lambda+O(a^{\mu+1}).\tag{14}$$

[5] To obtain a genuine Wick rotation and not just a discrete map, one introduces a complex parameter α in $l_t^2=-\alpha a^2$. The proper prescription leading to (12) is then an analytic continuation of α from 1 to -1 through the lower-half complex plane.

If the scaling is canonical (which means that the dimensionality of the renormalized coupling constant is the one expected from the classical theory), the exponent is given by $\mu = d$. Note that this construction requires a positive *bare* cosmological constant in order to make the state sum converge. Moreover, by virtue of relation (14) also the *renormalized* cosmological constant must be positive. Other than that, its numerical value is not determined by this argument, but by comparing observables of the theory which depend on Λ with actual physical measurements.[6] Another interesting observation is that the inclusion of a sum over topologies in the discretized sum (13) would lead to a super-exponential growth of at least $\propto N_d!$ of the number of triangulations with the volume N_d. Such a divergence of the path integral cannot be compensated by an additive renormalization of the cosmological constant of the kind outlined above. In the context of two-dimensional (Euclidean) quantum gravity this difficulty is well-known as the "absence of a physically motivated double-scaling limit" [21].

Lastly, obtaining an interesting continuum limit may or may not require an additional fine-tuning of the inverse gravitational coupling κ_{d-2}, depending on the dimension d. In four dimensions, one would expect to find a second-order transition along the critical line, corresponding to local gravitonic excitations. The situation in $d = 3$ is less clear, but results obtained so far indicate that no fine-tuning of Newton's constant is necessary [22, 23].

Before delving into the details, let me summarize briefly the results that have been obtained so far in the approach of Lorentzian dynamical triangulations. At the regularized level, that is, in the presence of finite cutoff a for the edge lengths and an infrared cutoff for large space-time volume, they are well-defined statistical models of Lorentzian random geometries in $d = 2, 3, 4$. In particular, they obey a suitable notion of reflection-positivity and possess selfadjoint Hamiltonians.

The crucial questions are then to what extent the underlying combinatorial problems of counting all d-dimensional geometries of a certain type can be solved, whether continuum theories with non-trivial dynamics exist and how their bare coupling constants get renormalized in the process. What we know about Lorentzian dynamical triangulations so far is that in dimension 2 and 3 they lead to continuum theories of quantum gravity. In $d = 2$, there is a complete analytic solution, which is distinct from the continuum theory produced by Euclidean dynamical triangulations. Also the matter-coupled model has been studied. In $d = 3$, there are numerical and partial analytical results which show that both a continuum theory exists and that it again differs from its Euclidean counterpart. Work on a more complete analytic solution which would give details about the geometric properties of the quantum theory is under way.

[6] The non-negativity of the renormalized cosmological coupling may be taken as a first "prediction" of our construction, which in the physical case of four dimensions is already in agreement with current observations.

In $d = 4$, the first numerical simulations are only just being set up. The challenge here is to do this for sufficiently large lattices, to be able to perform meaningful measurements. So far, we cannot make any statements about the existence and properties of a continuum theory in this physically most interesting case.

5.1 In two dimensions

The two-dimensional case serves as a nice illustration of the objectives of the approach, many of which can be carried out in a completely explicit manner [13]. There is just one type of building block, a flat Minkowskian triangle with two time-like edges of squared edge lengths $l_t^2 = -a^2$ and one space-like edge with $l_s^2 = a^2$. We build up a causal space-time from strips of unit height $\Delta t = 1$ (see Fig.7), where t is an integer-valued discrete parameter that labels subsequent spatial slices, ie. simplicial submanifolds of codimension 1 which are constructed from space-like links only. In the two-dimensional case these subspaces are one-dimensional. We choose periodic boundary conditions, such that the spatial "universes" are topologically spheres S^1 (other boundary conditions are also possible, leading to a slight modification of the effective quantum Hamiltonian [24, 25]). A spatial geometry at given t is completely characterized by its length $l(t) \in \{1, 2, 3, \ldots\}$, which (in units of the lattice spacing a) is simply the number of spatial edges it contains.

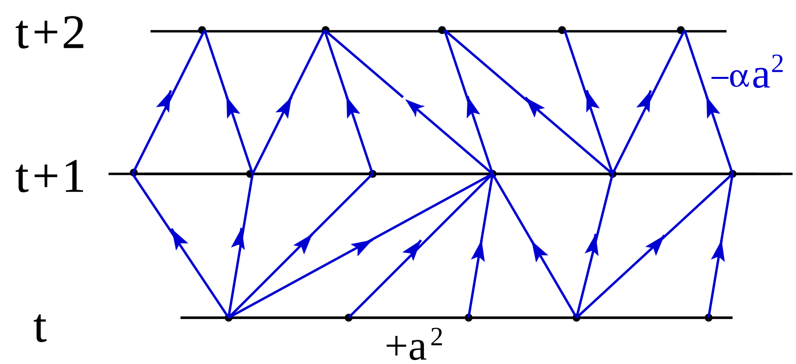

Figure 7: Two strips of a 2d Lorentzian triangulation, with spatial slices of constant t and interpolating future-oriented time-like links.

One simplification occurring in two dimensions is that the curvature term in the Einstein action is a topological invariant (that therefore does not depend on the metric), given by

$$\int_M d^2x \, \sqrt{|\det g|} R = 2\pi\chi, \tag{15}$$

where χ denotes the Euler characteristic of the two-dimensional space-time M. Since we are keeping the space-time topology fixed, the exponential of i times this term is a constant overall factor that can be pulled out from the path integral and hence does not contribute to the dynamics. Dropping this term, we can write the discrete path integral over 2d simplicial causal space-times as

$$G_\lambda(l_{\rm in}, l_{\rm out}; t) = \sum_{\substack{\text{causal } T\\ l_{\rm in}, l_{\rm out}, t}} \mathrm{e}^{-i\lambda N_2} \;\overset{\text{Wick}}{\longrightarrow}\; \sum_{\substack{W(T)\\ l_{\rm in}, l_{\rm out}, t}} \mathrm{e}^{-\tilde{\lambda} N_2}, \tag{16}$$

where the weight factors depend now only on the cosmological (volume) term, and $\tilde{\lambda}$ differs from λ by a finite positive numerical factor. Each history entering in the discrete propagator (16) has an in-geometry of length $l_{\rm in}$, an out-geometry of length $l_{\rm out}$, and consists of t steps. An important special case is the propagator for a single step, which in its Wick-rotated form reads[7]

$$G_{\tilde{\lambda}}(l_1, l_2; t=1) = \langle l_2 | \hat{T} | l_1 \rangle = \mathrm{e}^{-\tilde{\lambda}(l_1+l_2)} \sum_{T: l_1 \rightarrow l_2} 1 \equiv \mathrm{e}^{-\tilde{\lambda}(l_1+l_2)} \frac{1}{l_1+l_2} \begin{pmatrix} l_1+l_2 \\ l_1 \end{pmatrix}. \tag{17}$$

The second equation in (17) defines the transfer matrix $\hat{T}$ via its matrix elements in the basis of the (improper) length eigenvectors $|l\rangle$. Knowing the eigenvalues of the transfer matrix is tantamount to a solution of the general problem by virtue of the relation

$$G_{\tilde{\lambda}}(l_1, l_2; t) = \langle l_2 | \hat{T}^t | l_1 \rangle. \tag{18}$$

Importantly, the propagator satisfies the composition property

$$G_{\tilde{\lambda}}(l_1, l_2; t_1 + t_2) = \sum_{l=1}^{\infty} G_{\tilde{\lambda}}(l_1, l; t_1)\, l\, G_{\tilde{\lambda}}(l, l_2; t_2), \tag{19}$$

where the sum on the right-hand side is over a complete set of intermediate length eigenstates.

Next, we look for critical behaviour of the propagator $G_{\tilde{\lambda}}$ (that is, a non-analytic behaviour as a function of the renormalized coupling constant) in the limit as $a \rightarrow 0$. Since there is only one coupling, the phase diagram of the theory is just one-dimensional, and illustrated in Fig.8. As can be read off from the explicit form of the propagator,

$$G_{\tilde{\lambda}} = \sum_{N_2} \mathrm{e}^{-\tilde{\lambda} N_2} \sum_{T|_{N_2}} 1 = \sum_{N_2} \mathrm{e}^{-(\tilde{\lambda} - \tilde{\lambda}^{\rm crit}) N_2} \times \text{subleading}(N_2), \tag{20}$$

[7] This is the "unmarked" propagator, see [13, 11] for details.

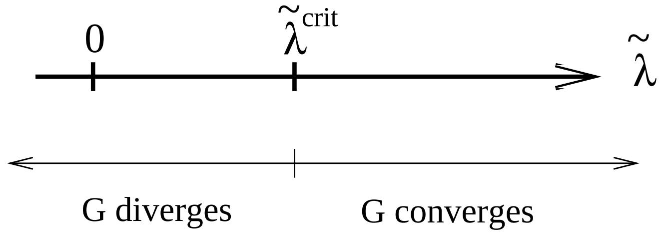

Figure 8: The 1d phase diagram of 2d Lorentzian dynamical triangulations.

the discrete sum over 2d geometries converges above some critical value $\tilde{\lambda}^{\text{crit}} > 0$, and diverges for $\tilde{\lambda}$ below this point. In order to attain a macroscopic physical volume $\langle V \rangle := \langle a^2 N_2 \rangle$ in the $a \to 0$ limit, one needs to approach $\tilde{\lambda}^{\text{crit}}$ from above. It turns out that to get a non-trivial continuum limit, the bare cosmological coupling constant has to be fine-tuned canonically according to

$$\tilde{\lambda} - \tilde{\lambda}^{\text{crit}} = a^2 \Lambda^{\text{ren}} + O(a^3). \tag{21}$$

Note that the numerical value of $\tilde{\lambda}^{\text{crit}}$ will depend on the details of the discretization (for example, the building blocks chosen; see [24] for alternative choices), the so-called non-universal properties of the model which do not affect the quantum dynamics of the final continuum theory. At the same time, the counting variables l and t are taken to infinity while keeping the dimensionful quantities $L := al$ and $T := at$ constant. The renormalized propagator is then defined as a function of all the renormalized variables,

$$G_\Lambda(L_1, L_2; T) := \lim_{a \to 0} a^\nu G_{\tilde{\lambda}^{\text{crit}} + a^2 \Lambda}\Big(\frac{L_1}{a}, \frac{L_2}{a}; \frac{T}{a}\Big), \tag{22}$$

which also contains a multiplicative wave function renormalization. The final result for the continuum path integral of two-dimensional Lorentzian quantum gravity is obtained by an inverse Wick rotation of the continuum proper time T to iT from the Euclidean expression and is given by

$$G_\Lambda(L_{\text{in}}, L_{\text{out}}; T) = \mathrm{e}^{-\coth(i\sqrt{\Lambda}T)\sqrt{\Lambda}(L_{\text{in}} + L_{\text{out}})} \, \frac{\sqrt{\Lambda L_{\text{in}} L_{\text{out}}}}{\sinh(i\sqrt{\Lambda}T)} \, I_1\left(\frac{2\sqrt{\Lambda L_{\text{in}} L_{\text{out}}}}{\sinh(i\sqrt{\Lambda}T)}\right), \tag{23}$$

where I_1 denotes the Bessel function of the first kind.

What is the physics behind this functional expression? In two dimensions, there is not much “physics” in the sense that the *classical* Einstein equations are empty. This renders meaningless the question of a classical limit of the 2d quantum theory; whatever dynamics there is will be purely “quantum”. Fig.9 shows a typical two-

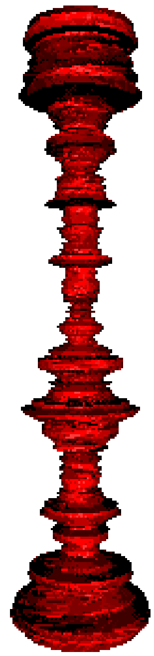

Figure 9: A typical two-dimensional Lorentzian space-time, at volume $N_2 = 18816$ and for a total proper time $t = 168$.

dimensional quantum universe: the compactified direction is “space”, and the vertical axis is “time”. It illustrates the typical development of the ground state of the system over time, as generated by a Monte-Carlo simulation of almost 19.000 triangles.

Since the theory has been solved analytically, we know the explicit form of the effective quantum Hamiltonian, namely,

$$\hat{H} = -L\frac{d^2}{dL^2} - 2\frac{d}{dL} + \Lambda L. \tag{24}$$

This operator is selfadjoint on the Hilbert space $L^2(\mathbf{R}_+, LdL)$ and generates a unitary evolution in the continuum proper time T. The Hamiltonian consists of a kinetic term in the single geometric variable L (the size of the spatial universe) and a potential term depending on the renormalized cosmological constant. Its spectrum is discrete,

$$E_n = 2(n+1)\sqrt{\Lambda}, \quad n = 0, 1, 2, \ldots \tag{25}$$

and one can compute various expectation values, for example,

$$\langle L \rangle_n = \frac{n+1}{\sqrt{\Lambda}}, \qquad \langle L^2 \rangle_n = \frac{3}{2}\frac{(n+1)^2}{\Lambda}. \tag{26}$$

Since there is just one dimensionful constant, with $[\Lambda] = \text{length}^{-2}$, all dimensionful quantities must appear in suitable units of Λ.

Another useful way of characterizing the continuum theory is via certain critical exponents, which in the case of gravitational theories are of a geometrical nature. The *Hausdorff dimension* d_H describes the scaling of the volume of a geodesic ball of radius R as a function of R. This very general notion can be applied to a fixed metric space, but for our purposes we are interested in the ensemble average over the entire "sum over geometries", that is, the leading-order scaling behaviour of the expectation value[8]

$$\langle V(R) \rangle \propto R^{d_H}. \tag{27}$$

The Hausdorff dimension is a truly dynamical quantity, and is *not* a priori the same as the dimensionality of the building blocks that were used to construct the individual discrete space-times in the first place. It may even depend on the length scale of the radial distance R. Remarkably, d_H can be calculated analytically in both Lorentzian and Euclidean 2d quantum gravity (see, for example, [26]). The latter, also known as "Liouville gravity", can be obtained by performing a sum over *arbitrary* triangulated Euclidean two-geometries (with fixed topology S^2), and not just those which correspond to a wick-rotated causal Lorentzian space-time. One finds

$$d_H = 2 \quad \text{(Lorentzian)} \qquad \text{and} \qquad d_H = 4 \quad \text{(Euclidean)}. \tag{28}$$

The geometric picture associated with the non-canonical value of d_H in the Euclidean case is that of a fractal geometry, with wildly branching "baby universes". This branching behaviour is incompatible with the causal structure required in the Lorentzian case, and the geometry of the quantum ground state is much better behaved, although it is by no means smooth as we have already seen.

We conclude that the continuum theories of 2d quantum gravity with Euclidean and Lorentzian signature are distinct. They can be related by a somewhat complicated renormalization procedure which one may think of as "integrating out the baby universes" [27], which is not at all as simple as "sticking a factor of i in the right place". In a way, this is not unexpected in view of the fact that (the spaces of) Euclidean and Lorentzian geometries are already classically very different objects. I am not claiming that from the point of view of 2d quantum gravity, one signature is better than the other. This seems a matter of taste, since neither theory describes any aspects of real nature. Nevertheless, what we have shown is that imposing causality constraints at the level of the individual histories in the path integral changes the outcome radically, a feature one may expect to generalize to higher dimensions.

[8]For the Lorentzian theory, "geodesic distance" refers to the length measurements after the Wick rotation.

Coming from Euclidean quantum gravity, there are specific reasons for looking at the behaviour of the matter-coupled theory in two dimensions. The coupling of matter fields to Lorentzian dynamical triangulations can be achieved in the usual manner by including for each given geometry T in the path integral a summation over all matter degrees of freedom on T, resulting in a double sum over geometric and matter variables. For example, adding Ising spins to 2d Lorentzian gravity is described by the partition function

$$Z(\lambda, \beta_I) = \sum_{N_2} \mathrm{e}^{-\lambda N_2} \sum_{\substack{\text{causal} \\ T \in \mathcal{T}_{N_2}}} \sum_{\{\sigma_i = \pm 1\}} \mathrm{e}^{\frac{\beta_I}{2} \sum_{<ij>} \sigma_i \sigma_j}, \tag{29}$$

where the last sum on the right is that of the Ising model on the triangulation T. The analogous model on Euclidean triangulations has been solved exactly [28], and its continuum matter behaviour is characterized by the critical exponents

$$\alpha = -1, \quad \beta = 0.5, \quad \gamma = 2, \qquad \text{(Euclidean)} \tag{30}$$

for the specific heat, the magnetization and the magnetic susceptibility respectively. These differ from the ones found for the Ising model on a fixed, flat lattice, the so-called Onsager exponents. The transition here is third-order, reflecting the influence of the fractal background on which the matter is propagating.

The same Ising model, when coupled to Lorentzian geometries according to (29), has not so far been solved exactly, but its critical matter exponents have been determined numerically and by means of a diagrammatic high-T expansion [29] and agree (within error bars) with the Onsager exponents, that is,

$$\alpha = 0, \quad \beta = 0.125, \quad \gamma = 1.75, \qquad \text{(Lorentzian)}. \tag{31}$$

So, interestingly, despite the fluctuations of the geometric ensemble evident in Fig.9, the conformal matter behaves *as if* it lived on a static flat lattice. This indicates a certain robustness of the Onsager behaviour in the presence of such fluctuations. Does it also imply there cannot be any back-reaction of the matter on the geometry? In order to answer this question, Lorentzian quantum gravity was coupled to "a lot of matter", in this case, eight copies of Ising models [30]. The partition function is a direct generalization of (29). For a given triangulation, there are 8 independent Ising models, which interact with each other only via their common interaction with the ensemble of geometries.

Looking again at a typical "universe", depicted in Fig.10, its geometry is now significantly changed in comparison with the case without matter. Part of it is squeezed down to a spatial universe of minimal size, with the remainder forming a genuinely extended space-time. A measurement of the critical behaviour of the matter on this

Figure 10: A typical two-dimensional Lorentzian geometry in the presence of eight Ising models, at volume $N_2 = 73926$ and for a total proper time $t = 333$.

piece of the universe again produces values compatible with the Onsager exponents![9] This is a very interesting result from the point of view of Liouville gravity, which does not seem to produce meaningful matter-coupled models beyond a central charge of one, the famous $c = 1$ barrier. (A model with n Ising spins corresponds to central charge $c = n/2$.) We conclude that *causal* space-times are more suitable carriers spaces for matter fields in 2d quantum gravity.

5.2 In three dimensions

Having discovered the many beautiful features of being Lorentzian in two dimensions, the next challenge is to solve the dynamically triangulated model in three dimensions and understand the geometric properties of the continuum theory it gives rise to. This will bring us a step closer to our ultimate goal, the four-dimensional quantum theory.

[9]The same would of course not hold for the degenerate part of the space-time which is effectively one-dimensional.

Despite its reputation as an "exactly soluble theory", many aspects of quantum gravity in 2+1 dimensions remain to be understood. There is still an unresolved tension between (i) the gauge (Chern-Simons) formulation in which the constraints can be solved in a straightforward way before or after quantization, leading to a quantized finite-dimensional phase space, and (ii) a path integral formulation in terms of "$g_{\mu\nu}$" which seems just about as intractable as the four-dimensional theory, and is power-counting non-renormalizable.

Since Lorentzian dynamical triangulations are really a regularized and non-perturbative version of the latter, a solution of the model should help to bridge this gap. Part of the trouble with gravitational path integrals is the "conformal-factor problem", which makes its first appearance in $d = 3$.[10] The conformal part of the metric, ie. the mode associated with an overall scaling of all components of the metric tensor, contributes to the action with a kinetic term of the wrong sign. This is most easily seen by considering just the curvature term of the Einstein action,

$$S = \int d^d x \sqrt{g}(R + \ldots), \tag{32}$$

and performing a conformal transformation $g_{\mu\nu} \to g'_{\mu\nu} = \mathrm{e}^{\phi} g_{\mu\nu}$ on the metric. This is *not* a gauge transformation and leads to a change

$$S \to S' = \int d^d x \sqrt{g'}(-(\partial_0 \phi)^2 + \ldots) \tag{33}$$

in the action, with the anticipated negative kinetic term for the conformal field ϕ. In the perturbative theory, this is not a real problem since the conformal term can be isolated explicitly and eliminated. However, the ensuing unboundedness of the action spells potential trouble for any non-perturbative geometric path integral (that is either Euclidean from the outset, or has been Euclideanized by a suitable Wick rotation), since the Euclidean weight factors $\exp(-S)$ $=\exp(\dot{\phi}^2 + \ldots)$ can become arbitrarily large. We will see that this problem arises in our approach too, and how it is resolved non-perturbatively.

First to some basics of Lorentzian dynamical triangulations in three dimensions. The construction of space-time manifolds is completely analogous to the 2d case. Slices of constant integer t are now two-dimensional space-like, equilateral triangulations of a given, fixed topology ${}^{(2)}\Sigma$, and time-like edges interpolate between adjacent slices t and $t+1$. The building blocks are given by two types of tetrahedra: one has three space-like and three time-like edges, and shares its space-like face with a slice $t =$const, the other has four time-like and two space-like edges, the latter belonging to two distinct adjacent spatial slices (Fig.11). We often denote the different tetrahedral types by the

[10] A more detailed account of the history of this problem in quantum gravity can be found in [19].

numbers of vertices (n, m) they have in common with two subsequent slices, which in three dimensions can take the values (3,1) (together with its time inverse (1,3)) and (2,2). Within a given sandwich $\Delta t = 1$, a (2,2)-tetrahedron can be glued to other (2,2)'s, as well as to (3,1)- and (1,3)-tetrahedra, but a (1,3) can never be glued directly to a (3,1), since their triangular faces do not match.

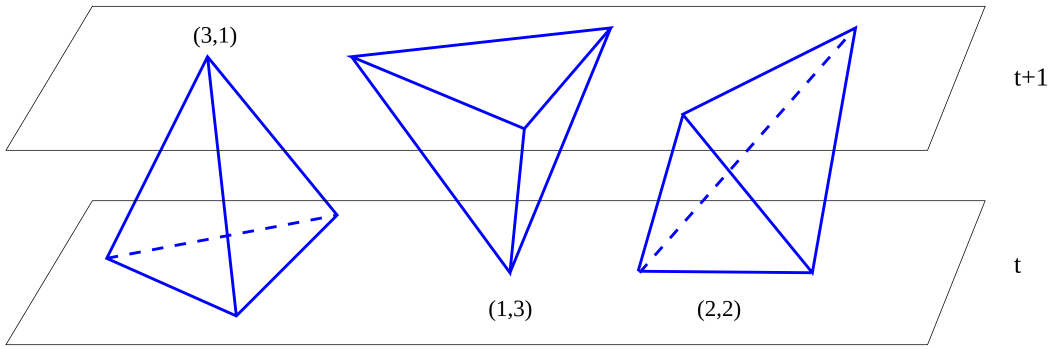

Figure 11: The three types of tetrahedral building blocks used in 3d Lorentzian gravity.

The simplicial action after the Wick rotation reads

$$S = -\kappa_1 N_1(T) + \kappa_3 N_3(T) \equiv N_3(T)\Big(-\kappa_1 \frac{N_1(T)}{N_3(T)} + \kappa_3\Big), \tag{34}$$

where the latter form is useful in the discussion of Monte-Carlo simulations, which are usually performed at (approximately) constant volume. The phase structure of the 3d model with spherical spatial topology, ${}^{(2)}\Sigma = S^2$, has been determined with the help of numerical simulations [22]. As expected, there is a critical line $\kappa_3^{\text{crit}}(\kappa_1)$. After fine-tuning to this line, there is no further phase transition[11] along it as a function of the inverse Newton coupling κ_1.

Where is our conformal-mode problem? If we keep the total volume N_3 fixed, the Euclidean action is not actually unbounded, but because of the nature of our regularization limited by the range of the "order parameter" $\xi := N_1/N_3$ which kinematically can only take values in the interval $[1, 5/4]$ [15]. This by no means implies we have removed the problem by hand. Firstly, one can explicitly identify configurations which minimize the action (34) and, secondly, the unboundedness could well be recovered in the continuum limit. However, what happens dynamically is that even in the continuum limit (as far as can be deduced from the simulations [22, 32]), ξ stays bounded away from its "conformal maximum", which means that the quantum theory of Lorentzian 3d gravity

[11] The first simulations did report a first-order transition at large κ_1, but this was presumably a numerical artefact; upon slightly generalizing the class of allowed geometries, this transition has now disappeared [31].

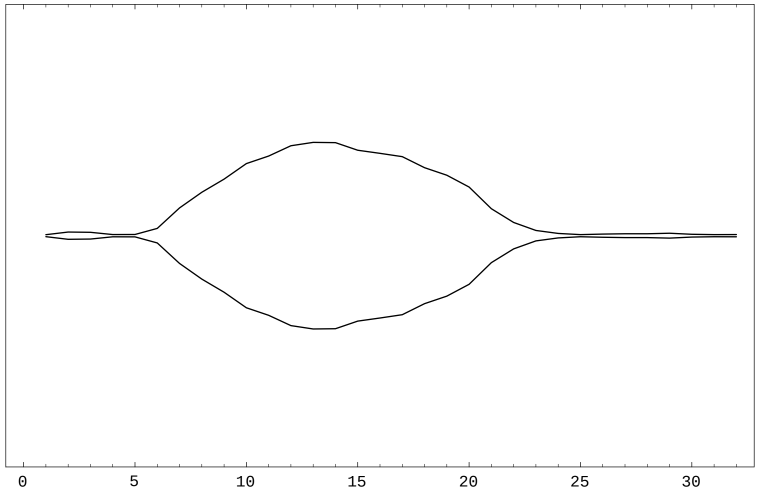

Figure 12: A typical three-dimensional universe, represented as a distribution of two-volumes $N_2(t)$ of spatial slices at proper times $t \in [0, 32]$, at $k_0 = 5.0$.

is *not* dominated by the dynamics of the conformal mode. Configurations with minimal action exist, but they are entropically suppressed. This is clearly a non-perturbative effect which involves not just the action, but also the "measure" of the path integral. A similar argument of a non-perturbative cancellation between certain Faddeev-Popov determinants and the conformal divergence can be made in a gauge-fixed continuum computation [19].[12]

This result is reassuring, because it shows that (Euclideanized) path integrals are not doomed to fail, if only they are set up properly and non-perturbatively. It also agrees with the expectation one has from canonical treatments of the theory where it is obvious that the conformal mode is not a propagating degree of freedom.

What can we say about the quantum dynamics of 3d Lorentzian gravity and the geometry of its ground state? Fig.12 shows a snapshot of a typical "universe" produced by the Monte-Carlo simulations. The only variable plotted as a function of the discrete time t is the two-volume of a spatial slice. What has been determined are the macroscopic scaling properties of this universe; they are in agreement with those of a genuine three-dimensional compact space-time, its time extent scaling $\propto N_3^{1/3}$ and its spatial volume $\propto N_3^{2/3}$.

Current efforts are directed at trying to analyze the detailed microscopic geometric properties of the quantum universe, its effective quantum Hamiltonian, and at gaining an explicit analytic understanding of the conformal-factor cancellation. How exactly

[12]Of course, the continuum path integral cannot really be *done*, so the cancellation argument relies on certain (plausible) assumptions about the behaviour of the path integral under renormalization.

does the conformal mode decouple from a propagator like $G(g^{(\mathrm{in})}, g^{(\mathrm{out})})$, although it appears among the labels parametrizing the in- and out-geometries g? – One would not in general expect to be able to make much progress in solving a three-dimensional statistical model analytically. However, we anticipate some simplifying features in the case of pure three-dimensional gravity, which is known to describe the dynamics of a finite number of physical parameters only.

There are two main strands of investigation, one for space-times $\mathbf{R}\times S^2$ and using matrix model techniques, and the other for space-times $\mathbf{R}\times T^2$ with flat toroidal spatial slices. An observation that is being used in both is the fact that the combinatorics of the transfer matrix, crucial to the solution of the full problem, is encoded in a *two*-dimensional graph. The transfer matrix $\hat{T}$, defined in analogy with (17), describes all possible transitions from one spatial 2d triangulation to the next. Such a transition is nothing but a three-dimensional sandwich geometry $[t, t+1]$, and is completely characterized by the two-dimensional pattern that emerges when one intersects this geometry at the intermediate time $t+1/2$. One associates with each time-like triangle a coloured edge where the triangle meets the $(t+1/2)$-surface. A blue edge belongs to a triangle whose base lies in the triangulation at time t, and a red edge denotes an upside-down triangle with base at $t+1$. The intersection pattern can therefore be viewed as a combined tri- and quadrangulation, made out of red triangles, blue triangles, and squares with alternating red and blue sides.

Graphs of this type, or equivalently their duals, are also generated by the large-N limit of a hermitian two-matrix model, with partition function

$$Z(\alpha_1,\alpha_2,\beta)=\int dA_{N\times N}\ dB_{N\times N}\ \mathrm{e}^{-N\ \mathrm{Tr}(\frac{1}{2}A^2+\frac{1}{2}B^2-\alpha_1 A^3-\alpha_2 B^3-\beta ABAB)}. \tag{35}$$

The cubic and quartic interaction terms in the exponent correspond to the tri- and four-valent intersections of the dual bi-coloured spherical graph characterizing a piece of space-time. In fact, as was shown in [33], the matrix model gives an embedding of the gravitational model we are after, since it generates *more* graphs than those corresponding to regular three-dimensional geometries. Interestingly, from a geometric point of view these can be interpreted as wormhole configurations. Some explicit examples are shown in Fig.13; the graphs consist of squares since they are taken from a "pyramid" variant of three-dimensional gravity, cf. footnote 13. Blue and red edges are in these pictures represented by solid and dashed lines.

The matrix model[13] has been solved analytically for the diagonal case $\alpha_1=\alpha_2$ [34], and its second-order phase transition separates the phase where wormholes are rare

[13]More precisely, a variant of (35) where the cubic terms A^3 and B^3 have been replaced by quartic terms A^4 and B^4. Geometrically, this corresponds to using pyramids instead of the tetrahedral building blocks, a difference that is unlikely to affect the continuum theory.

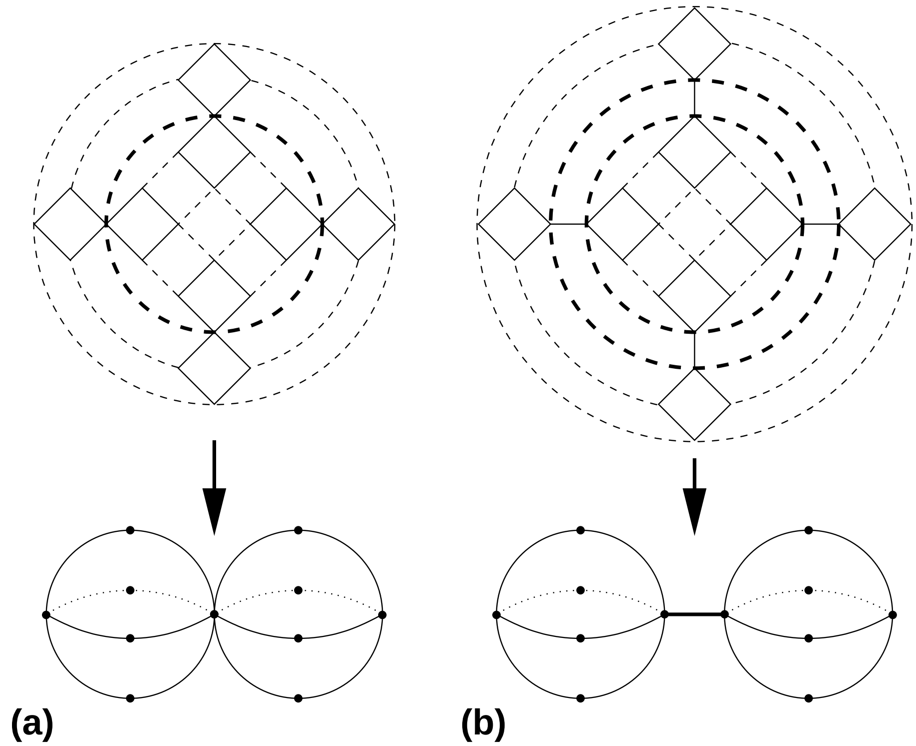

Figure 13: Examples of quadrangulations at $t + 1/2$ corresponding to wormholes at time t. Shrinking the dashed links to zero, one obtains the two-geometries at the bottom. The thick dashed lines at the top are contracted to points where wormholes begin or end.

from that where they are abundant. One therefore concludes that Lorentzian gravity as given by dynamical triangulations should correspond to the former.

It turns out that to extract information about the quantum Hamiltonian of the system, one must consider the off-diagonal case where the two α-couplings are different. Only in that case can one distinguish which part of the intersection graph comes from "below" (time t) and which one from "above" (time $t+1$). The colouring of the two-dimensional graph is really the memory of the original three-dimensional nature of the problem. It turns out that even for α's which differ only infinitesimally, this is a highly non-trivial problem. Making a natural ansatz for the analytic structure of the eigenvalue densities that appear in the partition function, a consistent set of equations has now been found, which will hopefully yield more details about the effective Hamiltonian of the quantum system [35]. Since there are no non-trivial Teichmüller parameters in the sphere case, what one might expect on dimensional grounds is a differential operator in the two-volume V_2 of the kind [31]

$$\hat{H} = c_1 V_2 \frac{d^2}{dV_2^2} - c_2 \Lambda V_2. \tag{36}$$

A second direction of attack are *cosmological models* of 3d gravity. They are symmetry-reduced in the sense that only a restricted class of spatial geometries is allowed at integer values of t, and also additional conditions may be imposed on the interpolating three-dimensional Lorentzian geometries. All models studied so far have flat tori as their spatial slices, the simplest case with a non-trivial physical configuration space, spanned by two real Teichmüller parameters. Flat two-dimensional tori can be obtained by suitably identifying the boundaries of a piece of the triangulated plane. Since we are working with equilateral triangles, this amounts to a piece of regular triangulation where exactly six triangles meet at every (interior) vertex point.

Even if the spatial slices have been chosen as spaces of constant curvature, this still leaves a number of possibilities of how the space-time in between can be filled in. One extreme choice would be to allow any intermediate three-geometry. By this we would probably not gain much in terms of simplifying the model, which obviously is a major motivation behind going "cosmological". By contrast, the first model studied had very simple interpolating geometries. The most transparent realization of this model is in terms of (4,1)- and (1,4)-pyramids rather than the (3,1)- and (1,3)-tetrahedra (a modification we already encoutered in the discussion of the matrix model), so that the spatial slices at integer-t are regular square lattices [36]. The corresponding 2d building blocks of the intersection graph at half-integer t are now blue squares, red squares and – as before – red-and-blue squares. If the (cut-open) tori at times $t_1 = t$ and $t_2 = t + 1$ consist of l_i columns and m_i rows, $i = 1, 2$, any allowed intersection pattern is a rectangle of size $(l_1 + l_2) \times (m_1 + m_2)$. An example is shown in Fig.14. The trouble with this simple model is that it does not have enough entropy: the number of possible interpolating sandwiches between two neighbouring spatial slices is given by

$$\text{entropy} \propto \binom{l_1 + l_2}{l_1} \binom{m_1 + m_2}{m_1}, \tag{37}$$

which is roughly speaking the square of the entropy of the two-dimensional Lorentzian model, cf. equation (17). This is not enough in the sense that the number of "microstates" in a piece of space-time $\Delta t = 1$ scales asymptotically only with the linear size of the tori, like $\exp(c{\cdot}\text{length})$. Such a behaviour cannot "compete" with the exponential damping $\exp(c'{\cdot}\text{area})$ coming from the cosmological term in the action. Thus, the only space-times that will not be exponentially damped in the continuum limit will be those whose spatial slices are essentially one-dimensional. This clearly is a limit

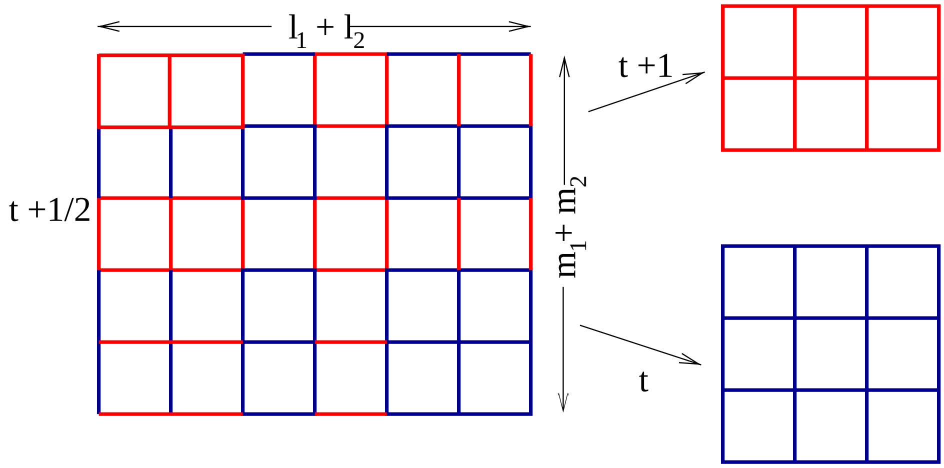

Figure 14: The cosmological “pyramid model” has regular slices at both integer and half-integer times.

that has nothing to do with the description of 3d quantum geometries we are after. In particular, the model is unsuitable for studying the conformal-mode cancellation.

I have included a discussion of this model because it suggests a potential problem for the path integral in models that impose severe symmetry constraints *before* quantization. Prime examples of this are mini-superspace models with only a finite number of dynamical degrees of freedom, whose path integral formulations are riddled with difficulties. Lorentzian dynamically triangulated models are more flexible concerning the imposition of such constraints. The next cosmological model I will consider has also flat tori at integer-t, but allows for more general geometries in between the slices. As a consequence, it does not suffer from the problem described above.

The easiest way of describing the geometry of this so-called *hexagon model* is by specifying the intersection patterns at half-integer t. One such pattern can be thought of as a tiling of a regular piece of a flat equilateral triangulation with three types of coloured rhombi. The colouring of the rhombi again encodes the orientation in three dimensions of the associated tetrahedral building block. A blue rhombus stands for a pair of (3,1)-tetrahedra, glued together along a common time-like face, a red rhombus for a pair of (1,3)-tetrahedra, and the rhombus with alternating blue and red sides is a (distorted) representation of a (2,2)-tetrahedron. Opposite sides of the regular triangular “background lattice” are to be identified to create the topology of a two-torus. The beautiful feature of this model is the fact that any complete tiling of this lattice by matching rhombic tiles *automatically* gives rise to flat two-tori on the two spatial boundaries of the associated sandwich $[t, t+1]$ [37].

After the Wick rotation, the one-step propagator of this model can be written as

$$G(g^{(1)}, g^{(2)}; \Delta t = 1) \equiv \langle g^{(2)} | \hat{T} | g^{(1)} \rangle = \mathcal{C}(g^{(1)}, g^{(2)})\ \mathrm{e}^{-S(g^{(1)}, g^{(2)})}. \tag{38}$$

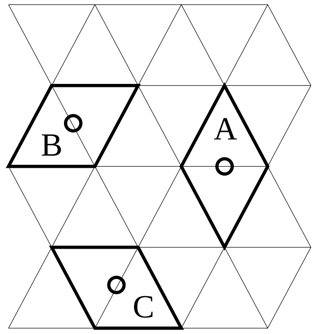

Figure 15: A rhombus can be put onto the triangular background lattice with three different orientations, A, B or C.

We note here a distinguishing property of the hexagon model, namely, a factorization of G into a Boltzmann weight $\exp(-S)$ and a combinatorial term $\mathcal{C}$ which counts the number of distinct sandwich geometries with fixed toroidal boundaries $g^{(1)}$ and $g^{(2)}$, both of which depend on the boundary data only, and not on the details of the three-dimensional triangulation of the interior. The leading asymptotics of the entropy term is determined by the combinatorics of a model of so-called *vicious walkers*. The walkers are usually represented by an ensemble of paths that move up a tilted square lattice, taking steps either diagonally to the left or to the right, in such a way that at most one path passes through any one lattice vertex.

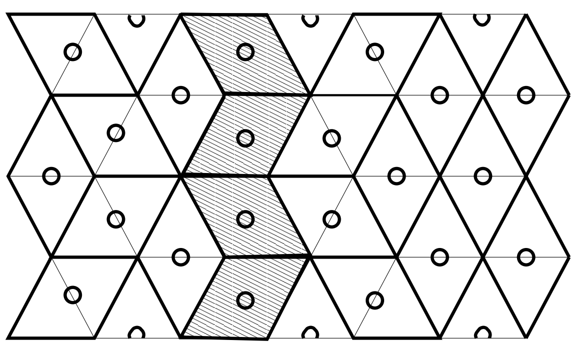

Figure 16: An example of a periodic tiling of the triangular background lattice. The shaded region is a B-C-path with winding number (0,1).

The paths of the hexagon model are sequences of rhombi that have been put down on the background lattice so they lie on one of their sides (types B and C in Fig.15). Because of the toroidal boundary conditions, such B-C-paths wind around the background lattice in the "vertical direction" (on figures such as Fig.16), which for the purposes of solving the 2d statistical model of vicious walkers we may think of as the time direction. The transfer matrix of this model can be diagonalized explicitly. Let us denote the number of vicious-walker paths by $w/2$, the width of the background lattice by $l+w$ and its height (in time direction) by m, all in lattice units. It turns out that for the simplest version of the model we can set $m = l$ without loss of generality. We are now interested in the number $\mathcal{N}(l, w)$ which solves the following combinatorial problem:

> Given two even integers l and w, how many ways $\mathcal{N}(l, w)$ are there of drawing $w/2$ non-intersecting paths of winding number $(0, 1)$ (in the horizontal and vertical direction) onto a tilted square lattice of width $l + w$ and height l, with periodic boundary conditions in both directions?

Denoting by $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{w/2})$, $\lambda_i \in \{0, 2, 4, \ldots, l + w \equiv 0\}$, the vector of positions of the vicious walkers along the horizontal axis, the eigenvectors of the transfer matrix have the form

$$\Psi(\vec{\lambda}) = \frac{1}{\sqrt{\frac{w}{2}!}} \det[z_j^{\lambda_i}], \qquad 1 \leq i, j \leq \frac{w}{2}, \tag{39}$$

where the complex numbers z_j are given by

$$z_j = \mathrm{e}^{i\pi \frac{k_j}{l+w}} \, \mathrm{e}^{i\pi \frac{w-2}{l+w}}, \qquad 0 \leq k_1 < k_2 \ldots < k_{w/2} \leq \frac{l+w}{2} - 1. \tag{40}$$

This result can be understood by observing that for a single walker in the same representation, taking a step to the right (left) is represented by a multiplication (division) by z, that is,

$$\Psi(\lambda) = z^{\lambda} \quad \Longrightarrow \quad z\Psi(\lambda) = z^{\lambda+1} \equiv \Psi(\lambda + 1). \tag{41}$$

The expression (39) is an appropriately antisymmetrized and normalized version for the case of several walkers. In this representation, the transfer matrix[14] takes the form

$$\hat{T}_{\mathrm{VW}} = \prod_{i=1}^{w/2} (\frac{1}{z_i} + 2 + z_i). \tag{42}$$

The final result in the limit as both $l, w \to \infty$, with a fixed ratio $\alpha := \frac{w}{l+w}$, is to leading order given by

$$\mathcal{N}(l, w) = C(\alpha)^{\frac{lw}{2}}, \qquad C(\alpha) = \exp\left[\frac{2}{\alpha} \int_0^{\alpha/2} dy \;\log(2\cos \pi y)\right]. \tag{43}$$

[14]This is the transfer matrix corresponding to a "double step" in time; a single step would lead to a position vector with odd λ_i's.

This shows that the hexagon model has indeed enough entropy, since the number of possible intermediate geometries scales exponentially with the area, and not just the linear dimension of the tori involved.

Another attractive feature of the model is that the Teichmüller parameters $\tau(t) = \tau_1(t) + i\tau_2(t)$ of the spatial tori at time t can be written explicitly as functions of the discrete variables describing the Lorentzian simplicial space-time. It turns out that the real parameter τ_1 is not dynamical, so that the wave functions of the model are labelled by just two numbers, the two-volume $v(t)$ and $\tau_2(t)$.[15] Expanding the euclideanized action for small $\Delta t = a$, one finds

$$S = \tilde{\lambda} v - \tilde{k} a^2 v \Big(\Big(\frac{\dot{v}}{v}\Big)^2 - \Big(\frac{\dot{\tau}_2}{\tau_2}\Big)^2 \Big) + \ldots, \tag{44}$$

where $\tilde{\lambda}$ and $\tilde{k}$ are proportional to the bare cosmological and inverse Newton's constants. This has the expected (and modular-invariant) form, with a standard kinetic term for τ_2, and one with the wrong sign for the area v. Of course, this is our old friend, the (global) conformal mode!

What we are after is the "effective action", containing contributions from both (44) and the state counting, namely,

$$S^{\rm eff} := S - \log(\text{entropy}) = v(\tilde{\lambda} - C) + \text{???} \tag{45}$$

In order to say anything about the cancellation or otherwise of the conformal divergence, we need more than just the leading-order term (43) of the entropy of the hexagon model. Unlike the exponential term, these subleading terms are sensitive to the colouring of the intersection graph, and efforts are under way to solve the corresponding vicious-walker problem [38].

5.3 Beyond three dimensions

As already mentioned earlier, there is nothing much to report at this stage on the nature of the continuum limit in the physical case of four dimensions. The first Monte-Carlo simulations are just set up, but any conclusive statements are likely to involve a combination of analytical and numerical arguments. Also it should be kept in mind that, unlike in previous simulations of four-dimensional *Euclidean* dynamical triangulations, the space-times involved here are not isotropic. Measurements of two-point functions, say, will be sensitive to whether the distances are time- or space-like, and therefore more computing power will be necessary to achieve a statistics comparable

[15] The model can be generalized to have non-trivial τ_1 by allowing for B-C-paths with higher winding numbers [38].

to the Euclidean case.

One way of making progress in four dimensions will be by studying geometries with special symmetries, along the lines of the 3d cosmological models discussed above. It should be noted that popular symmetry reductions, such as spherical or cylindrical symmetry, cannot be implemented *exactly* because of the nature of our discretization. They can at best be realized approximately, which in view of the results of the previous subsection may be a good thing since it will ensure that a sufficient number of microstates contributes to the state sum. An important application in this context is the construction of a path integral for spherical black hole configurations. Already the formulation of the problem has a number of challenging aspects, for example, the inclusion of non-trivial boundaries, an explicit realization of the (near-)spherical symmetry, and of a "horizon finder". Some of these problems have already been solved [39, 40]. It will be extremely interesting to see what Lorentzian dynamical triangulations have to say about the famous thermodynamic properties of quantum black holes from a non-perturbative point of view. These questions are currently under study.

6 Brief conclusion

As we have seen, the method of *Lorentzian dynamical triangulations* constitutes a well-defined regularized framework for constructing non-perturbative theories of quantum gravity. Technically, they can be characterized as regularized sums over simplicial random geometries with a time arrow and certain causality properties. In dimension $d < 4$, interesting continuum limits have been shown to exist. Their geometric properties have been explored, almost exhaustively in two, and partly in three dimensions. Both are examples of Lorentzian quantum gravitational theories which as continuum theories are *inequivalent* to their Euclidean counterparts, and the relation between the two is *not* that of some simple analytic continuation of the form $t \mapsto it$. The origin of the discrepancy between quantum gravity with Euclidean and Lorentzian signature lies in the absence of causality-violating branching points for geometries in the latter. Since in dimension $d \geq 3$, the approach of *Euclidean* dynamical triangulations seems to have serious problems, I am greatly encouraged by the fact that the 3d Lorentzian model is better behaved. Of course, it still needs to be verified explicitly that the imposition of causality conditions is indeed the correct remedy to cure the *four*-dimensional theory of its apparent diseases. One step in that direction will be to show that the non-perturbative cancellation mechanism for the conformal divergence is also present in $d = 4$.

Two warnings may be in order at this point. Firstly, there is a priori nothing *discrete* about the quantum gravitational theories this method produces. Its "discreteness" refers merely to the intermediate regularization that was chosen to make the non-perturbative path sums converge.[16] In particular, there is nothing in the construction suggesting the presence of any kind of “fundamental discreteness”, as has been found in canonical models of four-dimensional quantum gravity [41, 42, 43]. Secondly, one should refrain from trying to interpret the discrete expressions of the regularized model as some kind of approximation of the “real” quantum theory *before* one has shown the existence of a continuum limit which (at least in dimension four) describes an interacting theory of geometric degrees of freedom.

In conclusion, I have described here a possible path for constructing a non-perturbative quantum theory of gravity, by applying standard tools from both quantum field theory and the theory of critical phenomena to theories of fluctuating geometry. Investigation of the continuum theories in two and three space-time dimensions has already led to exciting new insights into the relation of the Lorentzian and Euclidean quantum theories, and ways of understanding and resolving the conformal sickness of gravitational path integrals, as well as bringing in new tools from combinatorics and statistical mechanics. I hope this has convinced you that the method of Lorentzian dynamical triangulations stands a good chance of throwing some light on the ever-elusive quantization of general relativity!

Acknowledgements. I would like to thank D. Giulini, C. Kiefer and C. Lämmerzahl for organizing an interesting and inspiring meeting at Bad Honnef, and for inviting me to speak there. Support through the EU network on “Discrete Random Geometry”, grant HPRN-CT-1999-00161, and the ESF network no.82 on “Geometry and Disorder” is also gratefully acknowledged.

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